Predicting dynamic behavior via anticipating synchronization in coupled pendulum-like systems

2009 J. Phys. A: Math. Theor. 42 335207

(http://iopscience.iop.org/1751-8121/42/33/335207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 335207 (15pp) doi:10.1088/1751-8113/42/33/335207

Predicting dynamic behavior via anticipating synchronization in coupled pendulum-like systems

Shiyun Xu and Ying Yang[1]

The State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University, Beijing 100871, People's Republic of China

E-mail: yy@mech.pku.edu.cn

Received 23 March 2009, in final form 27 June 2009
Published 30 July 2009
Online at stacks.iop.org/JPhysA/42/335207

Abstract

In this paper, the regime of anticipating synchronization (sometimes called predicted synchronization) in a class of nonlinear dynamical systems is investigated by testing the global asymptotical stability of time-delayed error dynamics. Sufficient conditions in terms of linear matrix inequalities are established for anticipating synchronization between such systems with and without state time delay. These results allow one to predict the dynamic behavior of the systems by using a copy of the same system that performs as a slave. Moreover, the cascaded anticipating synchronization is concerned such that several slave systems could anticipate the same master system with different delays. Concrete applications to phase-locked loops demonstrate the applicability and validity of the proposed results.

PACS numbers: 05.45.Xt, 05.45.−a

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Ever since the seminal work of Pecora and Carroll [1], there has been considerable interest in the phenomena of synchronization of complex or chaotic systems [2, 3]. Synchronization means that two or more systems adjust each other by coupling the systems or by forcing them, giving rise to common dynamical behavior [4]. The main motivation of synchronization so far, among others, seems to lie in potential applications regarding secure communications. Efforts have been made to characterize different types of synchronization in interacting chaotic systems. Complete synchronization, for example, implies coincidence of states of interacting systems, $y(t) = x(t)$, for all t larger than some transient time. Lag synchronization appears as

[1] Author to whom any correspondence should be addressed.

the occurrence of shifted-in-time states between two systems, namely $y(t) = x(t - \tau)$ with a certain positive constant $\tau > 0$ [5–7]. In contrast to lag synchronization, the peculiar phenomenon of anticipating synchronization [8, 9] shows that synchronization may possibly occur between master and time-delayed slave systems, and consequently in this manner the slave dynamics act as a predictor of the master dynamics in spite of the inherent unpredictability of chaotic systems and vice versa. The regime of anticipating synchronization, or sometimes called predicted synchronization, has been demonstrated through analytic and numerical evidence by considering a system of two coupled Ikeda equations, in a one-way delayed coupling configuration [8]:

$$\dot{x}(t) = -\alpha x(t) - \beta \sin x(t - \tau), \tag{1}$$

$$\dot{z}(t) = -\alpha z(t) - \beta \sin x(t), \tag{2}$$

where $x, z \in \mathbb{R}$ and the time delay $\tau > 0$. $\dot{e} = -\alpha e(t)$ is the error dynamics of $e(t) \overset{\triangle}{=} z(t - \tau) - x(t)$, and a necessary and sufficient condition for the error e to converge to 0 is that $\alpha > 0$. Therefore, system (2) synchronizes with the future states of (1) at time $t + \tau$ and hence anticipates the dynamics of (1). This regime has been demonstrated theoretically and numerically in disparate dynamical systems [10–14]. Experimental results have considered either electronic circuit implementations of the dynamical equations [15] or chaotic semiconductor diode lasers [16].

Although its main interest concerns systems whose dynamics has an intrinsic degree of unpredictability, the surprising result of anticipating synchronization is of general validity. In this regard, investigation on how anticipating synchronization can be achieved for general classes of complex or chaotic systems is of considerable interest to researchers. Except for the examples mentioned above, the anticipating synchronization for chaotic Lur'e systems has been specifically examined in virtue of the absolute stability theory [17, 18], which has been extensively studied based on the frequency-domain method and equivalent time-domain matrix inequality conditions. This kind of paradigm, however, hardly lends itself to the analysis of nonlinear systems with multiple equilibria. Regarded as a generalization of mathematical pendulum equations, pendulum-like systems are a typical category of multiple-equilibria nonlinear systems that have been observed in mechanical and electrical engineering applications such as synchronous machine and phase-locked loops (PLLs) [19]. Given appropriate system parameters and initial values, this kind of system could exhibit different kinds of nonlinear behavior, and sufficient conditions guaranteeing global properties of pendulum-like systems have been proposed based on both frequency-domain methods [20] and linear matrix inequality (LMI) [21] approaches in the time domain [22, 23]. As for pendulum-like systems with state delay, the global asymptotical stability has been examined in [24]. Thus, a natural and interesting question is: does the anticipating synchronization phenomenon exist between two coupled pendulum-like systems? Another question is: what scheme can be applied to and how can we determine the realization of this kind of synchronization problem?

Motivated by these considerations, this paper aims at examining the anticipating synchronization phenomenon in coupled pendulum-like dynamical systems, where one of the dynamical systems performs as the 'master system' while the remaining one acts as the 'slave'. Within the coupling setups considered here, which are generalizations of that considered in [8], coupling terms are acted upon in the slave one in order to synchronize the two systems. In the following contexts, the determination of anticipating synchronization is converted into an equivalent stabilizing problem for the error dynamics between master and slave systems, which is also shown as a standard form of a pendulum-like system. In order to find verifiable conditions guaranteeing anticipating synchronization, sufficient delay-dependent criteria in form of LMIs are derived by employing the Lyapunov–Krasovskii-type functionals. The solvability of such conditions can be tested through the Matlab LMI Toolbox. Consequently in this manner, the slave pendulum-like dynamics could act as a predictor of the master dynamics in spite of the inherent unpredictability of chaotic systems. Moreover, the scheme is extended to cascaded anticipating synchronization, which provides increased anticipation times [9], and thus several slave systems could anticipate the same master system at one time. It is found that the stability properties of the associated anticipatory synchronization manifold turn out to be the same as for identically synchronizing systems.

The rest of the paper is organized as follows. Preliminaries used are introduced in section 2. Section 3 presents the master–slave as well as cascaded anticipating synchronization schemes with criteria presented. Concrete examples showing the effectiveness and applicability of the proposed conditions have been given in section 4. Section 5 concludes the paper.

2. Preliminaries

Let us start out with the model of nonlinear pendulum-like dynamical system described by the following differential equations:

$$\dot{x} = Ax + B\varphi(y), \qquad \dot{y} = C^T x + D\varphi(y), \tag{3}$$

where $A \in \mathbb{R}^{n\times n}, B, C \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times m}$. Suppose that the parameter matrix pair (A, B) is controllable, (A, C^T) is observable and matrix A is Hurwitzian. The continuously differentiable vector-valued function $\varphi(y) = (\varphi_1(y_1), \ldots, \varphi_m(x_m))^T$ whose component $\varphi_l(y_l) : \mathbb{R} \to \mathbb{R}$ is T_l-periodic with a finite number of zeros on the interval $[0, T_l), l = 1, \ldots, m$. Also assume that $G(0)$ is nonsingular. The transfer function of the linear part from the input $\varphi(y)$ to the output $-\dot{y}$ is given by

$$G(s) = C^T (A - sI)^{-1} B - D.$$

Since the nonlinear function $\varphi(y)$ is T-periodic, the system possesses multiple isolated equilibria. The system with form (3) covers a wide class of systems in engineering, mechanics, electric and electronic circuits, and power systems [19].

Definition 1. *The pendulum-like system (3) is globally asymptotically stable if every solution* (x, y) *converges to a certain equilibrium* $(x_{\text{eq}}, y_{\text{eq}})$ *as t tends to* $+\infty$.

In practical situations, time delays caused by signal transmission affect the behavior of coupled systems; hence, to study the synchronization schemes between pendulum-like models with time delay is worth discussing. This study also considers the following system model with state time delay:

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau) + B\varphi(y(t)), \qquad \dot{y}(t) = C^T x(t) + D\varphi(y(t)), \tag{4}$$

where $A \in \mathbb{R}^{n\times n}, \tau \in \mathbb{R}$ is a constant time delay and other parameters are defined the same as in (3). The transfer function of the linear part from the input $\varphi(y)$ to the output $-\dot{y}$ is introduced as

$$G(s) = C^T (A + A_h \, e^{-hs} - sI)^{-1} B - D.$$

Suppose that $G(0) \neq 0$ and $A + A_h$ has no eigenvalues on the imaginary axis. Similarly, the pendulum-like model with state delay (4) also possesses multiple equilibria and the global asymptotical stability could also be depicted by definition 1.

3. Main results

In this section, we first examine two scenarios including the master–slave anticipation that generalize the scheme in (1) and (2) between two coupled identical pendulum-like systems with and without state time delay. Then a more generalized situation of cascaded anticipating synchronization is taken into account. These proposed approaches would be of significance to predict the dynamic behavior of pendulum-like systems using a copy of the same system that performs as a slave.

3.1. Master–slave anticipation

Consider two pendulum-like systems (3) coupled under the master–slave scheme with a master system subject to a nonlinear time-delay term and a slave one under control:

$$(\mathbf{M})\ \begin{cases} \dot{x}(t) = Ax(t) + B\varphi(\sigma_1(t)) + \Phi(\sigma_1(t-\tau)), \\ \dot{\sigma_1}(t) = C^T x(t) + D\varphi(\sigma_1(t)), \end{cases} \tag{5}$$

$$(\mathbf{S})\ \begin{cases} \dot{y}(t) = Ay(t) + B\varphi(\sigma_2(t)) + \Phi(\sigma_1(t)) + u(t), \\ \dot{\sigma_2}(t) = C^T y(t) + D\varphi(\sigma_2(t)), \end{cases} \tag{6}$$

where the master and slave systems are pendulum-like systems with state vectors $x, y \in \mathbb{R}^n$ and phase vectors $\sigma_1, \sigma_2 \in \mathbb{R}^m$, respectively. Constant matrices A, B, C, D as well as the T-periodic function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ are defined the same as in (3). Φ is a nonlinear term such that solutions of the above systems exist. $u(t)$ is a unidirectionally coupled term, which is supposed to be in the form of

$$u(t) = K(y(t-\tau) - x(t)),$$

with a certain positive constant $\tau > 0$ and the coupling matrix $K \in \mathbb{R}^{n\times n}$.

By defining the error signals $e(t) = x(t) - y(t-\tau)$, $\xi(t) = \sigma_1(t) - \sigma_2(t-\tau)$, we obtain the error dynamics

$$\begin{cases} \dot{e}(t) = Ae(t) + Ke(t-\tau) + B\eta(\xi(t), \sigma_2(t)), \\ \dot{\xi}(t) = C^T e(t) + D\eta(\xi(t), \sigma_2(t)), \end{cases} \tag{7}$$

in which $\eta(\xi(t), \sigma_2(t)) = \varphi(\xi(t) + \sigma_2(t-\tau)) - \varphi(\sigma_2(t-\tau))$.

Definition 2. *If the error dynamics (7) is globally asymptotically stable, then the master system and the slave one are said to achieve anticipating synchronization.*

Remark 1. Due to the periodicity of function $\varphi(\cdot)$, nonlinearity $\eta(\xi, \sigma_2)$ is also a T-periodic function with respect to ξ, and the error dynamics (7) still possesses a standard form of pendulum-like system with multiple equilibria. As a consequence, if the error dynamical system (7) is globally asymptotically stable, then each of its solutions tends to a certain equilibrium as $t \to \infty$, that is, $e(t) \to 0, \xi \to c$. Here, c is a certain constant satisfying $\eta(c, \sigma_2) = 0$, which is determined by the initial conditions $\xi(0)$ and $\sigma_2(0)$. In this regard, when the master and slave systems achieve anticipating synchronization, the state variables $x(t) \to y(t-\tau)$, while the phase variable $\sigma_1(t) \to \sigma_2(t-\tau) + c$ as $t \to \infty$.

Remark 2. Alternatively one could consider a unidirectionally coupling term $u(t) = K(y(t-\tau_1) - x(t-\tau_2))$, with $\tau_1 > 0$ and $\tau_2 > 0$ in (6). Synchronization would be achieved whenever $x(t-\tau_2) - y(t-\tau_1)$ tends to 0 as t tends to ∞. If $\tau_1 = \tau_2$, the synchronization would be complete; the anticipating (respectively lag) synchronization would be achieved if

$\tau_1 - \tau_2 > 0$ (respectively $\tau_1 - \tau_2 < 0$). The introduction of such a coupling term is the approach considered in [25], where techniques similar to ours are used. Nevertheless, their model is notably different from ours. In [25], if the coupling term u is suppressed then both systems are identical and uncoupled. Hence, there always exists an invariant manifold (synchronization manifold) where $x(t - \tau_2) = y(t - \tau_1)$. This is not our case and only when $\tau_1 - \tau_2 > 0$ do we get a synchronization manifold.

The main objective of the synchronization scheme (5) and (6) is to let the slave system **(S)** anticipate the dynamics of the master system **(M)** by adding a coupling term to the slave system **(S)**, specifically to ensure global asymptotical stability of the error dynamical system (7) in virtue of an LMI-based approach to the determination of a feedback controller for such a property.

The following theorem presents a criterion formulated by means of LMI to ensure that the controlled slave system could anticipate the future behavior of the master system.

Theorem 1. *If there are positive definite matrices* $W > 0, Z > 0, Y > 0$, *matrices* M, N, P, Q, R, S *and diagonal matrices* $\kappa, \epsilon > 0, \delta > 0$ *such that the following LMIs hold:*

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & M - A^T N^T + W + S^T & -\bar{\tau} P \\ * & \Pi_{22} & \Pi_{23} & -K^T N^T - S^T & -\bar{\tau} Q \\ * & * & \Pi_{33} & -B^T N^T & -\bar{\tau} R \\ * & * & * & \bar{\tau} Z + \mathrm{He}\, N & -\bar{\tau} S \\ * & * & * & * & -\bar{\tau} Z \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} 2\epsilon & \kappa\nu \\ * & 2\delta \end{bmatrix} > 0, \tag{9}$$

with

$$\Pi_{11} = Y + \mathrm{He}\,(P - MA) + C\epsilon C^T, \qquad \Pi_{12} = -P + Q^T - MK,$$
$$\Pi_{22} = -\mathrm{He}\, Q - Y, \qquad \Pi_{13} = \tfrac{1}{2} C\kappa + R^T - MB + C\epsilon D,$$
$$\Pi_{23} = -R^T, \qquad \Pi_{33} = \delta + \tfrac{1}{2}\mathrm{He}\, D^T\kappa + D^T\epsilon D,$$

then for any delay τ *satisfying* $0 < \tau \leqslant \bar{\tau}$, *the error dynamics (7) is globally asymptotically stable, and the master system* **(M)** *and the slave one* **(S)** *achieve anticipating synchronization.*

Proof. See the appendix. □

Remark 3. In virtue of theorem 1, one could easily handle the controller design problem and further the anticipating synchronization between the two systems. Moreover, several slack variables M, N, P, Q, R, S are introduced into LMI (8) by employing the Newton–Leibniz formula. It is thus expected that theorem 1 will be less conservative than some existing results due to the increasing freedom of these slack variables [26, 27].

Remark 4. When the parameter matrix K of the coupling term is unknown, we could design an appropriate coupling matrix in order to realize the anticipating synchronization between the master and slave systems. However, the matrix inequality (8) would become nonlinear since K acts as a variable. Under this circumstance, we could specify $M = \alpha I$ and $N = \beta I$, where α and β are numbers to be searched. Then we have the following LMI condition for the design of K.

Corollary 1. *If there are positive definite matrices $W > 0, Z > 0, Y > 0$, matrices P, Q, R, S, K, diagonal matrices $\kappa, \epsilon > 0, \delta > 0$ and two numbers α, β such that LMI*

$$\begin{bmatrix} \Psi & -P+Q^T-\alpha K & \frac{1}{2}C\kappa+R^T-\alpha B+C\epsilon D & \alpha I-\beta A^T+W+S^T & -\bar{\tau}P \\ * & -\mathrm{He}\,Q-Y & -R^T & -\beta K^T-S^T & -\bar{\tau}Q \\ * & * & \delta+\frac{1}{2}\mathrm{He}\,D^T\kappa+D^T\epsilon D & -\beta B^T & -\bar{\tau}R \\ * & * & * & \bar{\tau}Z+2\beta I & -\bar{\tau}S \\ * & * & * & * & -\bar{\tau}Z \end{bmatrix} < 0 \tag{10}$$

*as well as (9) holds, then for any delay τ satisfying $0 < \tau \leqslant \bar{\tau}$, the master system **(M)** and the slave one **(S)** achieve anticipating synchronization under the derived coupling matrix K, where $\Psi = Y + \mathrm{He}(P-\alpha A) + C\epsilon C^T$ and α, β are numbers to be searched.*

3.2. Anticipating synchronization between systems with state time delay

In what follows, consider a more general case that the master and slave systems are both nonlinear pendulum-like systems with state time delays as presented in (4):

$$(\widetilde{\mathbf{M}})\begin{cases}\dot{x}(t) = Ax(t)+A_dx(t-\tau)+B\varphi(\sigma_1(t))+\Phi(\sigma_1(t-\tau)),\\ \dot{\sigma_1}(t) = C^Tx(t)+D\varphi(\sigma_1(t)),\end{cases}$$
$$(\widetilde{\mathbf{S}})\begin{cases}\dot{y}(t) = Ay(t)+A_dy(t-\tau)+B\varphi(\sigma_2(t))+\Phi(\sigma_1(t))+u(t),\\ \dot{\sigma_2}(t) = C^Ty(t)+D\varphi(\sigma_2(t)),\end{cases} \tag{11}$$

where $A_d \in \mathbb{R}^{n\times n}$, and other parameters are defined the same as in systems (5) and (6). By introducing the coupling term $u(t) = K(y(t-\tau)-x(t))$, we can derive the error dynamics

$$\begin{cases}\dot{e}(t) = Ae(t)+(A_d+K)e(t-\tau)+B\eta(\xi,\sigma_2)\\ \dot{\xi}(t) = C^Te(t)+D\eta(\xi,\sigma_2).\end{cases} \tag{12}$$

with the error signals $e(t) = x(t)-y(t-\tau)$ and $\xi(t) = \sigma_1(t)-\sigma_2(t-\tau)$. Nonlinearity $\eta(\xi,\sigma_2) = \varphi(\xi(t)+\sigma_2(t-\tau))-\varphi(\sigma_2(t-\tau))$ is a periodic function of ξ due to the periodicity of function $\varphi(\cdot)$; hence, the error dynamics (12) also possesses a standard form of pendulum-like system. Accordingly, the criterion derived for anticipating synchronization of (5) and (6) could be directly applied to systems $(\widetilde{\mathbf{M}})$ and $(\widetilde{\mathbf{S}})$. Based on the Schur lemma, theorem 1 can be converted to the following equivalence, whose proof is omitted.

Theorem 2. *If there are positive definite matrices $W > 0, Z > 0, Y > 0$, matrices M, N, P, Q, R, S and diagonal matrices $\kappa, \epsilon > 0, \delta > 0$ such that*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & M-A^TN^T+W+S^T & -\bar{\tau}P & C\epsilon \\ * & \Xi_{22} & \Xi_{23} & -A_d^TN^T-K^TN^T-S^T & -\bar{\tau}Q & 0 \\ * & * & \Xi_{33} & -B^TN^T & -\bar{\tau}R & D^T\epsilon \\ * & * & * & \bar{\tau}Z+\mathrm{He}\,N & -\bar{\tau}S & 0 \\ * & * & * & * & -\bar{\tau}Z & 0 \\ * & * & * & * & * & -\epsilon \end{bmatrix} < 0 \tag{13}$$

and (9) hold, then for any delay τ satisfying $0 < \tau \leqslant \bar{\tau}$, the master system $(\widetilde{\mathbf{M}})$ and the slave one $(\widetilde{\mathbf{S}})$ with state time delay in (11) achieve anticipating synchronization, where

$$\Xi_{11} = Y+\mathrm{He}(P-MA), \qquad \Xi_{12} = -P+Q^T-MA_d-MK,$$
$$\Xi_{22} = -\mathrm{He}\,Q-Y, \qquad \Xi_{13} = \tfrac{1}{2}C\kappa+R^T-MB,$$
$$\Xi_{23} = -R^T, \qquad \Xi_{33} = \delta+\tfrac{1}{2}\,\mathrm{He}\,D^T\kappa.$$

3.3. Cascaded anticipation

In this part, we consider the situation when the slave systems in (6) are a chain of l pendulum-like dynamical systems ($\mathbf{S_i}$), $i = 1, \ldots, l$:

$$
\begin{aligned}
&(\mathbf{M}) \begin{cases} \dot{x} = Ax + B\varphi(\sigma) + \Phi(\sigma(t-\tau)), \\ \dot{\sigma} = C^T x + D\varphi(\sigma), \end{cases} \\
&(\mathbf{S_1}) \begin{cases} \dot{y}_1 = Ay + B\varphi(\sigma_1) + \Phi(\sigma(t)) + u_1(t), \\ \dot{\sigma_1} = C^T y_1 + D\varphi(\sigma_1), \end{cases} \\
&(\mathbf{S_2}) \begin{cases} \dot{y}_2 = Ay_2 + B\varphi(\sigma_2) + \Phi(\sigma_1(t)) + u_2(t), \\ \dot{\sigma_2} = C^T y_2 + D\varphi(\sigma_2), \end{cases} \\
&\qquad\qquad\vdots \\
&(\mathbf{S_l}) \begin{cases} \dot{y}_l = Ay_l + B\varphi(\sigma_l) + \Phi(\sigma_{l-1}(t)) + u_l(t), \\ \dot{\sigma_l} = C^T y_l + D\varphi(\sigma_l), \end{cases}
\end{aligned}
\tag{14}
$$

where $u_1(t) = K(y_1(t-\tau) - x(t)), u_2(t) = K(y_2(t-2\tau) - y_1(t-\tau)), \ldots, u_l(t) = K[y_l(t-l\tau) - y_{l-1}(t-(l-1)\tau)]$. Suppose that the nonlinearity Φ is T_Φ-periodic, where $T = nT_\Phi$ with integer $n > 0$. Similar to definition 2, the following definition of cascaded anticipating synchronization is given in allusion to the $l+1$ pendulum-like systems (14).

Definition 3. *Systems (14) are said to be cascaded anticipating synchronized, if for any constant $\tau > 0$, state variables of the master system and l slave systems achieve $x(t) = y_1(t-\tau) = y_2(t-2\tau) = \cdots = y_l(t-l\tau)$ as time goes to infinity, while the phase variables $\sigma(t) = \sigma_i(t - \mathrm{i}\tau) + c_i$, in which $c_i = \delta_i T$ are constants for $\delta_i \in \mathbb{Z}, i = 1, \ldots, l$.*

The idea of applying cascaded synchronization was initially proposed by Voss in the context of anticipating synchronization between coupled ordinary differential equation systems [9]. This concept is of significance, in that it provides increased anticipation times for the anticipating synchronization phenomenon. Theorem 3 presents a sufficient condition on cascaded anticipating synchronization among systems (14).

Theorem 3. *If there are positive definite matrices $W > 0, Z > 0, Y > 0$, matrices M, N, P, Q, R, S and diagonal matrices $\kappa, \epsilon > 0, \delta > 0$ such that LMIs (8) and (9) hold, then for any delay τ satisfying $0 < l\tau \leqslant \bar{\tau}$, the master pendulum-like dynamical system* **(M)** *and the slave ones* ($\mathbf{S_i}$)*, $i = 1, \ldots, l$, achieve cascaded anticipating synchronization.*

Proof. Define the error signal $e_1(t) = x(t) - y_1(t-\tau)$ and $\xi_1(t) = \sigma(t) - \sigma_1(t-\tau)$; then the dynamics of synchronization error between the master and slave systems can be represented as

$$
\begin{cases} \dot{e_1}(t) = Ae_1(t) + Ke_1(t-\tau) + B\eta(\xi_1, \sigma_1), \\ \dot{\xi}_1(t) = C^T e_1(t) + D\eta(\xi_1, \sigma_1). \end{cases} \tag{15}
$$

Here, $\eta(\xi_1, \sigma_1) = \varphi(\xi_1(t) + \sigma_1(t-\tau)) - \varphi(\sigma_1(t-\tau))$, and it is clear that (15) has the same form as (7); hence, the criterion derived in theorem 1 ensures that systems **(M)** and ($\mathbf{S_1}$) could achieve anticipating synchronization, namely $x(t) = y_1(t-\tau)$ as $t \to \infty$.

The result of anticipating synchronization between **(M)** and ($\mathbf{S_2}$) is obtained by undertaking the investigation of the error signals $e_2(t) = x(t) - y_2(t-2\tau), \xi_2(t) = \sigma(t) - \sigma_2(t-2\tau)$ and error dynamics

$$
\begin{cases} \dot{e_2}(t) = Ae_2(t) + Ke_2(t-2\tau) + B\eta(\xi_2, \sigma_2) + [\Phi(\sigma_1(t-2\tau)) - \Phi(\sigma(t-\tau))] \\ \dot{\xi}_2(t) = C^T e_2(t) + D\eta(\xi_2, \sigma_2) \end{cases}
$$

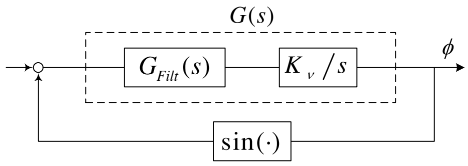


Figure 1. A simplified version of the PLL model.

with $\eta(\xi_2, \sigma_2) = \varphi(\xi_2(t) + \sigma_2(t - 2\tau)) - \varphi(\sigma_2(t - 2\tau))$. Given that anticipating synchronization between **(M)** and **(S_1)** has already taken place, we have $\Phi(\sigma_1(t - 2\tau)) - \Phi(\sigma_1(t-2\tau)+\delta_1 T) = \Phi(\sigma_1(t-2\tau)) - \Phi(\sigma_1(t-2\tau)+n\delta_1 T_\Phi) = 0$. Accordingly, we arrive at the error dynamics

$$\begin{cases} \dot{e}_2(t) = Ae_2(t) + Ke_2(t - 2\tau) + B\eta(\xi_2, \sigma_2), \\ \dot{\xi}_2(t) = C^T e_2(t) + D\eta(\xi_2, \sigma_2). \end{cases} \tag{16}$$

It is obvious that the difference between the two error dynamics (15) and (16) merely embodies in the doubled constant time delay. Based on the proof of theorem 1, (8) and (9) ensure $x(t) = y_1(t - \tau) = y_2(t - 2\tau)$ for any $0 < 2\tau \leqslant \bar{\tau}$ as $t \to \infty$. By analogy, if the LMIs (8) and (9) are feasible for the constant time delay $l\tau$, then all of the l error dynamics would be ensured to be asymptotically stable, which means that the master system **(M)** and the chain of l pendulum-like dynamical systems **(S_i)**, $i = 1, \ldots, l$, have achieved cascaded anticipating synchronization. This completes the proof. □

Remark 5. Theorem 3 provides an effective criterion for the verification of cascaded anticipating synchronization. It could be observed that under the circumstance of cascaded anticipating synchronization, the anticipation times has increased. Also, the stability properties of the associated anticipatory synchronization manifold turn out to be the same as for identically synchronizing systems as shown in theorem 1.

4. Illustrative examples

The phase-locked loop (PLL) is a kind of concrete system studied in the theory of communication [20, 28] whose dynamics is described by (3) after certain simplifications [29]. In the same manner, a group of two interconnected PLLs could be treated as master–slave-coupled pendulum-like systems. Many monographs have been devoted to the extremely significant locking-in property for an arbitrary solution to an equilibrium point which is looked upon as a synonym of the global asymptotical stability [30, 31]. The simplified model of the PLL is shown in figure 1, and it will be shown how results derived in this paper can be used to predict the dynamical behavior of such systems. The transfer function from the input sin(·) to the output $-\phi$ is given by $G(s) = \frac{1}{s}\chi(s)$, where $\chi(s)$ is the transfer function from sin(·) to the output $-\dot{\phi}$ and described by the state space model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(\rho(t)) + \Phi(\rho(t - \tau)), \\ \dot{\rho}(t) &= Cx(t) + D\varphi(\rho(t)), \end{aligned} \tag{17}$$

with the nonlinear function $\varphi(\rho(t)) = \sin(\rho(t))$, and

$$A = \begin{bmatrix} 0.1 & -0.4 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix},$$

$$D = -0.5,\ \Phi(\rho(t-\tau)) = \begin{bmatrix} 0.6\sin(0.5\rho(t-\tau)) \\ 0 \end{bmatrix}.$$

Such a system describes the dynamics of an autonomous phase-locked loop with a second-order filter of type '2/2' and a 2π input nonlinearity [20].

Example 1. The anticipating synchronization between two linearly coupled identical systems **(M)** and **(S)** with parameters as (17) is to be examined in this example. Suppose the time-delayed coupling term $u(t) = K(y(t-\tau) - x(t))$ is added to the master system. To verify our results, a suitable K is determined as $K = \mathrm{diag}(-0.14, -1.52)$ and the constant time delay $\tau = 1$. This representation results in $m = 1$, and clearly A has no eigenvalues on the imaginary axis and (A, B) is controllable. In this case, the LMIs (8) and (9) are tested to be feasible with

$$W = 10^3 \times \begin{bmatrix} 0.8368 & -0.1586 \\ -0.1586 & 1.1880 \end{bmatrix}, \qquad k = 649.5551,$$

$$Y = \begin{bmatrix} 127.5612 & -47.5088 \\ -47.5088 & 480.1620 \end{bmatrix}, \qquad e = 171.6160,$$

$$Z = \begin{bmatrix} 505.0932 & -17.9090 \\ -17.9090 & 404.1529 \end{bmatrix}, \qquad d = 78.6019,$$

which implies that the master and slave systems (5) and (6) realize anticipating synchronization according to theorem 1. Furthermore, we get the largest allowable upper bound of the time delay $\tau^* = 1.7356$, which guarantees that the master and slave systems will achieve anticipating synchronization for any τ subject to $0 < \tau \leqslant 1.7356$.

This estimation could be further demonstrated through graphical illustrations. Simulation results for the master–slave synchronization scheme with initial values $[x(0), \sigma_1(0)] = [0.1\ -1.2\ -0.52]$ and $[y(0), \sigma_2(0)] = [-0.3\ -0.21\ 3]$ are given in figure 2, which shows that the dynamics of the master system could be exactly predicted by the slave one. The insets (right) in figure 1 are the zooms revealing the anticipation effect, while the prediction error dynamics $e_i(t) = x_i(t) - y_i(t-\tau), i = 1, 2$, and $\xi(t) = \sigma_1(t) - \sigma_2(t-\tau)$ are depicted in the insets (left), respectively, showing asymptotic convergence to zero as $t \to \infty$.

Example 2. The cascaded anticipating synchronization is illustrated in this example. Consider a master PLL system **(M)** and two slave ones **(S$_1$)**, **(S$_2$)** that are connected under the frame of (14). Suppose that the system parameters are the same as those in example 1, except for the nonlinear term $\Phi(\rho(t-\tau)) = \begin{bmatrix} 0.6\sin(2\rho(t-\tau)) \\ 0 \end{bmatrix}$. In this regard, $T = 2T_\Phi = 2\pi$, which satisfy the assumption imposed on (14). Pick the constant time delay as $\tau = 0.5$. Due to the feasibility of LMIs (8) and (9) with $\tau^* = 1.7356$, the conditions in theorem 3 hold for time delays $\tau = 0.5$ and $2\tau = 1$, which guarantees that cascaded anticipation could be achieved. For the purpose of demonstration, the simulation results with initial values $[x(0), \sigma(0)] = [0.1\ -1.2\ -0.52]$, $[y_1(0), \sigma_1(0)] = [-0.3\ -0.21\ 3]$ and $[y_2(0), \sigma_2(0)] = [1.1\ -2.3\ -2.5]$ are given in figure 3, with error dynamics $e_{i1}(t), e_{i2}(t)$ and $\xi_i(t)$, respectively, $i = 1, 2$, depicted in the corresponding insets. This illustrative result coincides with theorem 3 and confirms the achievement of cascaded anticipating synchronization.

5. Conclusion

In summary, we have considered the master–slave anticipating synchronization of two pendulum-like dynamical systems by dealing with a time-delayed nonlinear system with

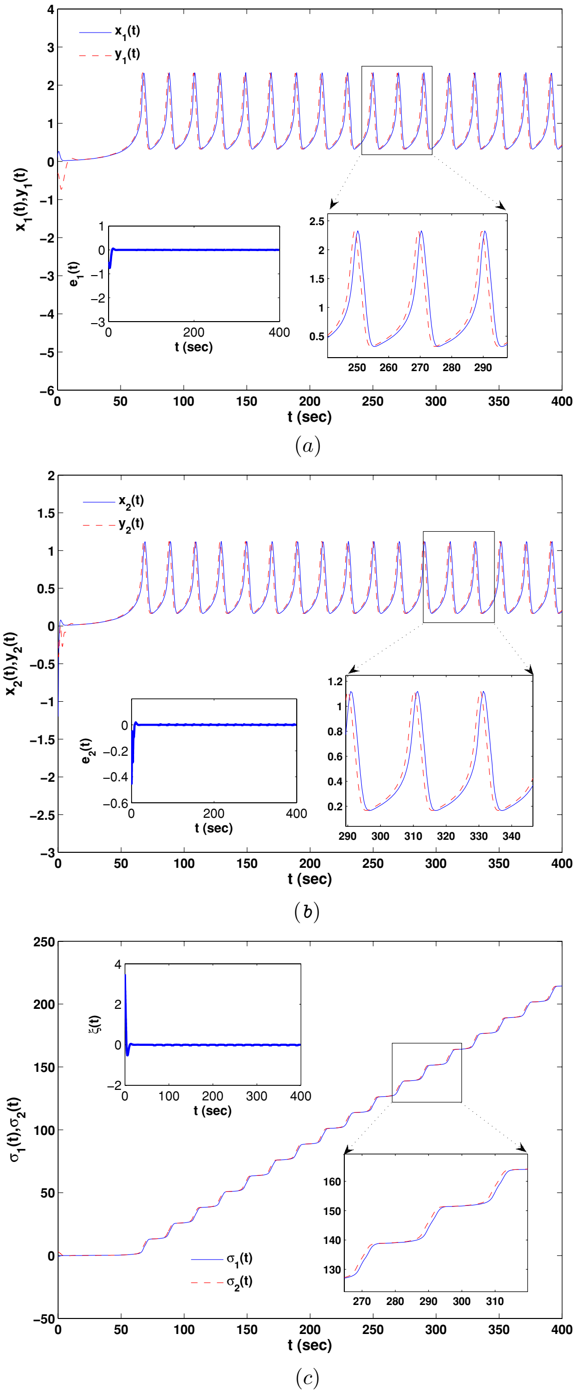
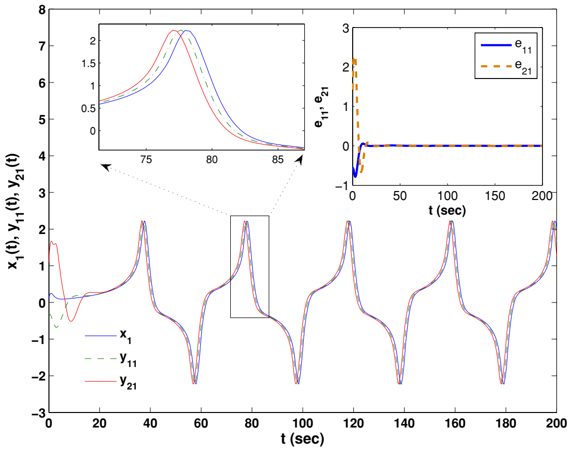
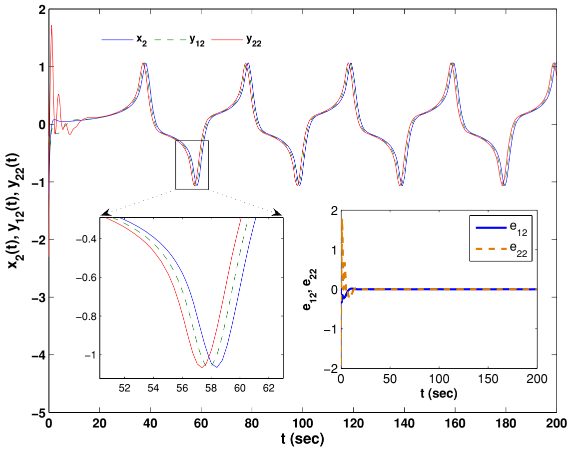


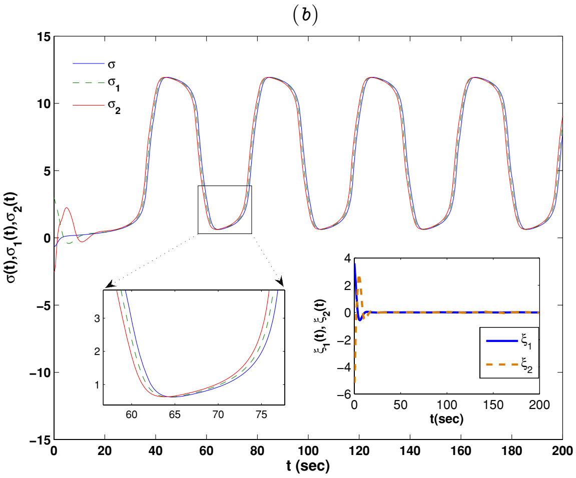
Figure 2. Anticipating synchronous dynamics of (5) and (6): (*a*) $x_1(t)$ and $y_1(t)$, (*b*) $x_2(t)$ and $y_2(t)$, (*c*) $\sigma_1(t)$ and $\sigma_2(t)$.



(*a*)



(*b*)



(*c*)

Figure 3. Anticipating synchronous dynamics of (14): (*a*) $x_1(t)$ and $y_{i1}(t)$; (*b*) $x_2(t)$ and $y_{i2}(t)$; (*c*) $\sigma(t)$ and $\sigma_i(t)$; $i = 1, 2$.

multiple equilibria. A sufficient condition in the form of LMI was established to guarantee the anticipating synchronization of such systems, which are applicable to predict states of possibly chaotic or complex systems with time delay. The derived result was then extended to that of cascaded anticipating synchronization among a number of slave pendulum-like systems. Throughout this paper, the model is assumed to be exactly known, which is rather difficult to achieve in a practical situation. Based on practical consideration, the anticipating synchronization problem could be generalized to systems with model uncertainties and perturbation, which will be the subject of a future study.

Acknowledgments

The authors would like to thank the anonymous reviewer for giving constructive comments on this paper. This work is supported by the National Science Foundation of China under grant 60874011.

Appendix. Proof of theorem 1

The following lemma will be of great help as a proof of theorem 1.

Lemma 1 [20]. *Suppose that both* $\alpha : \mathbb{R}_+ \to \mathbb{R}$ *and* $\beta : \mathbb{R}_+ \to \mathbb{R}$ *belong to* $L^2[0, +\infty)$*; then for* $t \to +\infty$*, we have the following statement:*

$$\gamma(t) = \int_0^t \alpha(t-\tau)\beta(\tau)\,\mathrm{d}\tau \to 0.$$

Before moving on further, define a diagonal matrix $\nu = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_m)$ with

$$\nu_k = \frac{\int_0^{T_k} \eta_k(\xi_k, \sigma_{2k})\,\mathrm{d}\eta}{\int_0^{T_k} |\eta_k(\xi_k, \sigma_{2k})|\,\mathrm{d}\eta}, \qquad k = 1, 2, \ldots, m, \tag{A.1}$$

together with a function $F_k(\xi_k) = \eta_k(\xi_k, \sigma_{2k}) - \nu_k|\eta_k(\xi_k, \sigma_{2k})|$, which satisfies that $\int_0^{T_k} F_k(\xi)\,\mathrm{d}\xi = 0$.

Proof of theorem 1. Construct a Lyapunov function candidate as

$$V(e, \xi) = e^T W e + \int_{-\tau}^0 \int_{t+b}^t \dot{e}^T(a) Z \dot{e}(a)\,\mathrm{d}a\,\mathrm{d}b + \int_{t-\tau}^t e^T(a) Y e(a)\,\mathrm{d}a + \sum_{k=1}^m \kappa_k \int_0^{\xi_k} F_k(y)\,\mathrm{d}y.$$

It is known from the Newton–Leibniz formula [32] that

$$e(t-\tau) = e(t) - \int_{t-\tau}^t \dot{e}(a)\,\mathrm{d}a.$$

Denote $e = e(t), e_\tau = e(t-\tau)$ and $\eta = \eta(\xi, \sigma_2)$ for simplicity. For any matrix $\Upsilon_1 = [M^T\ N^T]$, $\Upsilon_2 = [P^T\ Q^T\ R^T\ S^T]$, it arrives at the following formulation:

$$\Omega_1 = [e^T \quad \dot{e}^T]\Upsilon_1^T[\dot{e} - Ae - Ke_\tau - B\eta] = 0,$$

$$\Omega_2 = [e^T \quad e_\tau^T \quad \eta^T \quad \dot{e}^T]\Upsilon_2^T\left[\int_{t-\tau}^t \dot{e}(a)\,\mathrm{d}a + e_\tau - e\right] = 0,$$

where M, N, P, Q, R, S are matrices with appropriate dimensions. By incorporating the term 2Ω, the time derivative of $V(e, \xi)$ along any trajectory of system (7) is derived as

$$\begin{aligned}
\dot{V}(e,\xi) &= 2e^T W\dot{e} + \int_{-\tau}^{0} [\dot{e}^T(t)Z\dot{e}(t) - \dot{e}^T(t+b)Z\dot{e}(t+b)]\,\mathrm{d}b \\
&\quad + \frac{1}{\tau}\int_{t-\tau}^{t} \left(e^T Y e - e_\tau^T Y e_\tau\right)\mathrm{d}a + \sum_{k=1}^{m} \kappa_k F_k(\xi_k)\dot{\xi}_k + 2\Omega_1 + 2\Omega_2 \\
&= \frac{2}{\tau}\int_{t-\tau}^{t} \left[e^T(W+S^T)\dot{e} + e^T P e + e^T(-P+Q^T)e_\tau + e^T(R^T)\eta + e_\tau^T(-Q)e_\tau\right. \\
&\quad + e_\tau^T(-R^T)\eta + \dot{e}^T(-S)e_\tau + e^T(-\tau P)\dot{e}(a) + e_\tau^T(-\tau Q)\dot{e}(a) + \eta^T(-\tau R)\dot{e}(a) \\
&\quad \left.+ \dot{e}^T(-\tau S)\dot{e}(a)\right]\mathrm{d}a + e^T(Y - MA - A^T M^T)e + \frac{1}{\tau}\int_{t-\tau}^{t} [\dot{e}^T(t)\tau Z\dot{e}(t) \\
&\quad - \dot{e}^T(a)\tau Z\dot{e}(a)\,\mathrm{d}a + \frac{1}{\tau}\int_{t-\tau}^{t} \left[2e^T(M - A^T N^T)\dot{e} + 2e^T(-MK)e_\tau + 2e^T(-MB)\eta\right. \\
&\quad \left.+ 2\dot{e}^T N\dot{e} + 2\dot{e}^T(-NK)e_\tau + 2\dot{e}^T(-NB)\eta - e_\tau^T Y e_\tau\right]\mathrm{d}a + 2\dot{e}^T(-NA)e \\
&\quad + \sum_{k=1}^{m}\left[\epsilon_k\dot{\xi}_k^2 + \delta_k\eta_k^2(\xi_k,\sigma_{2k})\right] + \sum_{k=1}^{m}\left[\kappa_k\eta_k(\xi_k,\sigma_{2k})\dot{\xi}_k - \kappa_k\nu_k|\eta_k(\xi_k,\sigma_{2k})|\dot{\xi}_k\right. \\
&\quad \left.- \epsilon_k\dot{\xi}_k{}^2 - \delta_k\eta_k^2(\xi_k,\sigma_{2k})\right].
\end{aligned}$$

Observe that condition (9) is equivalent to the existence of two positive numbers $\delta_{0k} > 0$ and $\epsilon_{0k} > 0$ such that

$$\kappa_k\nu_k|\eta_k(\xi_k,\sigma_{2k})|\dot{\xi}_k + \epsilon_k\dot{\xi}_k{}^2 + \delta_k\eta_k^2(\epsilon_k,\sigma_{2k}) \geqslant \epsilon_{0k}\dot{\xi}_k^2 + \delta_{0k}\eta_k^2(\epsilon_k,\sigma_{2k});$$

then we arrive at

$$\begin{aligned}
\dot{V}(\theta,\xi) + \sum_{k=1}^{m}\left[\epsilon_{0k}\dot{\xi}_k^2 + \delta_{0k}\eta_k^2(\epsilon_k,\sigma_{2k})\right] &\leqslant \dot{V}_1(\theta,\xi) + \dot{V}_2(\theta,\xi) + \dot{V}_3(\theta,\xi) \\
&\quad + \sum_{k=1}^{m}\left[\epsilon_k\dot{\xi}_k^2 + \kappa_k\eta_k(\xi_k,\sigma_{2k})\dot{\xi}_k + \delta_k\eta_k^2(\xi_k,\sigma_{2k})\right] = \frac{1}{\tau}\int_{t-\tau}^{t}\chi\Pi\chi^T\,\mathrm{d}a,
\end{aligned} \tag{A.2}$$

where $\chi = \left[e^T e_\tau^T \eta^T \dot{e}^T \dot{e}^T(a)\right]$ and Π is defined in (8). By integrating both sides of (A.2), it is ensured by the LMIs (8) and (9) that

$$V(t) - V(0) \leqslant -\sum_{k=1}^{m}\int_0^t \left[\epsilon_{0k}\dot{\xi}_k^2 + \delta_{0k}\eta_k^2(\xi_k,\sigma_{2k})\right]\mathrm{d}t. \tag{A.3}$$

Due to the stability of A, we know that solution $e(t)$ of (7) is bounded [23], while the boundedness of $\int_0^{\xi_k} F_k(y)\,\mathrm{d}y$ is ensured by the fact that $F_k(\xi_k)$ have mean value zero. It follows that $V(e,\xi)$ is bounded. Together with (A.3), we get

(a) $\int_0^{+\infty}\eta_k^2(\xi_k(t),\sigma_{2k}(t))\,\mathrm{d}t < +\infty$,
(b) $\int_0^{+\infty}\dot{\xi}_k^2(t)\,\mathrm{d}t < +\infty$,
(c) $\int_0^{+\infty}e_k^2(t)\,\mathrm{d}t < +\infty, k = 1, 2, \ldots, m$.

It is known that if the solutions of (7) are bounded, then the nonlinear functions $\eta_k(\xi_k,\sigma_{2k}), k = 1, 2, \ldots, m$, are uniformly continuous on $[0, +\infty)$ [20]. Also, since η_i are periodic functions and have a finite number of zeros on $[0, T_i)$, we have

$$\begin{aligned}
&\lim_{t\to+\infty}\eta_k(\xi_k(t),\sigma_{2k}(t)) = 0, \\
&\lim_{t\to+\infty}\xi_k(t) = \hat{\xi}_k(t), \qquad k = 1, 2, \ldots, m,
\end{aligned} \tag{A.4}$$

with $\eta_k(\hat{\xi}_k(t), \sigma_{2k}(t)) = 0$. Now let us consider a solution of the error dynamics $e(t) = \mathbf{e}^{At}e(0) + \int_0^t \mathbf{e}^{A(t-s)}Ke(s-\tau)\,\mathrm{d}s + \int_0^t \mathbf{e}^{A(t-s)}B\eta(s)\,\mathrm{d}s$. It can be deduced from lemma 1 as well as conditions (a) and (c) that $\lim_{t\to+\infty}\mathbf{e}^{-Ah}\int_0^t \mathbf{e}^{A(t-s)}Be(s)\,\mathrm{d}s = 0$. Combining with the Hurwitzian stability of matrix A, we have

$$\lim_{t\to+\infty} e(t) = 0. \tag{A.5}$$

The validity of the second equality in (A.4) and (A.5) shows that every solution $(e(t), \xi(t))$ of system (7) converges to a certain equilibrium, which indicates that the error dynamics (7) is globally asymptotically stable, namely the LMIs given in (8) and (9) could ensure that the master system (**M**) and slave system (**S**) achieve anticipating synchronization. □

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821–4
[2] Boccaletti S, Kurths J, Osipov G, Valladares D L and Zhou C S 2002 The synchronization of chaotic systems *Phys. Rep.* **366** 1–101
[3] Wu C W 2002 *Synchronization in Coupled Chaotic Circuits and Systems* (Singapore: World Scientific)
[4] Chen G and Dong X 1998 *From Chaos to Order: Methodologies, Perspectives and Applications* (Singapore: World Scientific)
[5] Rosenblum M G, Pikovsky A S and Kurths J 1997 From phase to lag synchronization in coupled chaotic oscillators *Phys. Rev. Lett.* **78** 4193–6
[6] Shahverdiev E M, Sivaprakasam S and Shore K A 2002 Lag synchronization in time-delayed systems *Phys. Lett.* A **292** 320–4
[7] Li C D, Liao X F and Wong K W 2004 Chaotic lag synchronization of coupled time-delayed systems and its applications in secure communication *Physica* D **194** 187–202
[8] Voss H U 2000 Anticipating chaotic synchronization *Phys. Rev.* E **61** 5115–9
[9] Voss H U 2001 Dynamic long-term anticipation of chaotic states *Phys. Rev. Lett.* **87** 014102
[10] Masoller C 2001 Anticipation in the synchronization of chaotic semiconductor lasers with optical feedback *Phys. Rev. Lett.* **86** 2782–5
[11] Hernández-Garcá E, Masoller C and Mirasso C R 2002 Anticipating the dynamics of chaotic maps *Phys. Lett.* A **295** 39–43
[12] Ciszak M, Calvo O, Masoller C, Mirasso C and Toral R 2003 Anticipating the response of excitable systems driven by random forcing *Phys. Rev. Lett.* **90** 204102
[13] Toral R, Masoller C, Mirasso C R, Ciszak M and Calvo O 2003 Characterization of the anticipated synchronization regime in the coupled FitzHugh–Nagumo model for neurons *Physica* A **325** 192–8
[14] Xu S Y, Yang Y and Song L 2009 Control-oriented approaches to anticipating synchronization of chaotic deterministic ratchets *Phys. Lett.* A **373** 2226–36
[15] Voss H U 2002 Real-time anticipation of chaotic states of an electronic circuit *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **12** 1619–25
[16] Sivaprakasam S, Shahverdiev E M, Spencer P S and Shore K A 2001 Experimental demonstration of anticipating synchronization in chaotic semiconductor lasers with optical feedback *Phys. Rev. Lett.* **87** 154101
[17] Huijberts H, Nijmeijer H and Oguchi T 2007 Anticipating synchronization of chaotic Lur'e systems *Chaos* **17** 013117
[18] Oguchi T and Nijmeijer H 2005 Prediction of chaotic behavior *IEEE Trans. Circuits Syst.* I **52** 2464–72
[19] Leonov G A, Reitmann V and Smirnova V B 1992 *Non-Local Methods for Pendulum-Like Feedback Systems* (Teubner: Stuttgart-Leipzig)
[20] Leonov G A, Ponomarenko D V and Smirnova V B 1996 *Frequency-Domain Methods for Nonlinear Analysis* (Singapore: World Scientific)
[21] Boyd S, ElGhaoui L, Feron E and Balakrishnam V 1994 *Linear Matrix Inequalities in Systems and Control* (Philadelphia, PA: SIAM)
[22] Yang Y and Huang L 2003 H_∞ controller synthesis for pendulum-like systems *Syst. Control Lett.* **50** 263–76
[23] Duan Z S, Wang J Z and Huang L 2007 Criteria for dichotomy and gradient-like behavior of a class of nonlinear systems with multiple equilibria *Automatica* **43** 1583–9
[24] Lu P L, Yang Y and Huang L 2007 Global asymptotic stability of pendulum-like system with state delay *Proc. 46th IEEE Conf. on Decision and Control* pp 2229–34

[25] Huang X and Cao J D 2006 Generalized synchronization for delayed chaotic neural networks: a novel coupling scheme *Nonlinearity* **19** 2797–811

[26] He Y, Wu M, She J H and Liu G P 2004 Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays *Syst. Control Lett.* **51** 57–65

[27] Wu M, He Y, She J H and Liu G P 2004 Delay-dependent criteria for robust stability of time-varying delay systems *Automatica* **40** 1435–9

[28] Gardner F M 1979 *Phaselock Techniques* (New York: Wiley)

[29] Yang Y, Fu R and Huang L 2004 Robust analysis and synthesis for a class of uncertain nonlinear systems with multiple equilibria *Syst. Control Lett.* **53** 89–105

[30] Abramovitch D Y 1990 Lyapunov redesign of analog phase-lock loops *IEEE Trans. Commun.* **38** 2197–202

[31] Buckwalter J and York R A 2000 Time delay considerations in high-frequency phase-locked loop *IEEE Radio Frequency Integrated Circuits Symp.* pp 181–4

[32] Xu S Y and Lam J 2005 Improved delay-dependent stability criteria for time-delay systems *IEEE Trans. Autom. Control* **50** 384–7